

# Dicke superradiance, Bose-Einstein condensation of photons and spontaneous symmetry breaking

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## Abstract

It is shown that the phenomenon of Dicke superradiance essentially occurs due to spontaneous symmetry breaking. Two generalised versions of the Dicke model are studied, and compared with a model that describes photonic Bose-Einstein condensate, which was experimentally realised sometime back. In all the models, it is seen that, the occurrence of spontaneous symmetry breaking is responsible for coherent radiation emission.

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The phenomenon of Bose-Einstein condensation (BEC) was proposed by Einstein in 1924, who showed that a noninteracting bosonic gas can undergo a phase transition below a critical temperature to macroscopically populate the ground state [1]. Although in dilute atomic gases it was observed more than a decade back, its manifestations in different physical phenomena, like superfluidity, is known since a long time [2–4]. Owing to the fact that, the condensation phenomenon leads to coherence at macroscopic scale in a quantum system, makes it a subject of intense investigation.

Since the phenomenon of BEC is a property of bosonic statistics, one wonders whether it could be possible to have BEC of photons. In the vacuum, owing to vanishing chemical potential it is not possible to have BEC. However as shown by Chiao, if the photons are trapped in a cavity, then it would be possible to have BEC of them [5]. In 2010, Klaers *et. al.* managed to experimentally realise BEC of photons, using an optical micro-cavity filled with a nonlinear dye, along these lines [6]. Since BEC leads to macroscopic occupation of a particular cavity mode, it led to coherent radiation emission from the cavity, which was seen in the experiment.

In 1953, Dicke proposed a model, wherein a collection of two-level atoms are interacting resonantly with a single cavity mode [7, 8]. It was found that when the system is in, what is known as, the superradiant state, the emission of photons is coherent. Later, it was found that, the system undergoes a phase transition below a critical temperature to yield the superradiant phase [9, 10]. The phenomenon of superradiance has been and continues to be a subject of interest, and has been realised experimentally in different systems [8, 11–15].

Since both BEC of photons, as observed by Klaers *et. al.*, and Dicke superradiance lead to coherent emission of photons, one wonders whether there exist any relation between the physics governing the two effects. The aim of this paper is to study these effects in two different models, and to show that the phenomenon of spontaneous symmetry breaking (SSB) is responsible for coherent radiation emission in both the systems. The phenomenon of BEC in a model that, describes the photon dynamics in the experiment of Klaers *et. al.* is briefly reviewed. It is shown how the occurrence of condensation of photons, presence of optical coherence and spontaneous symmetry breaking are intertwined. The phenomenon of superradiance is studied by solving two generalisations of the Dicke model, wherein a single cavity mode is resonantly interacting with a collection of three-level and with four-level atoms, respectively. The partition function of both these models is found, and the

phase structure of these systems is studied. It is found that below a critical temperature, the both the systems undergo a phase transition to yield the superradiant state, leading to coherent photon emission. The connection between the superradiant phase, BEC and SSB is discussed.

## I. BOSE-EINSTEIN CONDENSATION AND SPONTANEOUS SYMMETRY BREAKING

Consider a nonrelativistic theory of a quantum field  $\psi$ , describing the dynamics of non-interacting bosons. The equation of motion for such a theory is:

$$\left(i\frac{\partial}{\partial t} + \nabla^2\right)\psi = 0. \quad (1)$$

Note that the equation of motion is invariant under the field redefinition:  $\psi \rightarrow \psi e^{i\theta}$ , and hence such a freedom of field redefinition is said to be the symmetry of the theory. Often the theory is defined such that, the vacuum is required to be annihilated by  $\psi$ :  $\psi|0\rangle = 0$ . This implies that the ground state of the system has no particles, since number operator expectation value is vanishing:  $\langle 0|N|0\rangle = \int dV \langle 0|\psi^\dagger\psi|0\rangle = 0$ . However, suppose that the theory realises a ground state  $|vac\rangle$  which is such that:  $\psi|vac\rangle = c|vac\rangle$ ,  $c$  being a constant. Then it is clear that the system in the ground state has infinitely many particles, since expectation value of particle number density is finite:  $\langle vac|N|vac\rangle/V = |c|^2 \neq 0$ . This means that there is a macroscopic occupation of particles in a particular state, which is nothing but BEC [1]. Note that the symmetry  $\psi \rightarrow \psi e^{i\theta}$  is not respected by the ground state, since  $\langle vac|\psi|vac\rangle = c$  is fixed. Another way to view this fact is as follows. The phase symmetry  $\psi \rightarrow \psi e^{i\theta}$  is generated by  $N$ , the unitary transformation which does this transformation is given by:  $U = e^{i\theta N}$ , so that:  $U^{-1}\psi U = \psi e^{i\theta}$ . However the action of  $N$  on  $|vac\rangle$  is ill-defined, since  $\langle vac|N|vac\rangle \rightarrow \infty$ , in the thermodynamic limit. This phenomenon is called spontaneous symmetry breaking [16, 17]. Thus one says that the theory has realised a ground state in which, the phase invariance is spontaneously broken. It must be emphasised that, the field redefinition freedom is never actually lost, and a careful study shows that there appears gapless Nambu-Goldstone modes in such cases in the system, which maintain the symmetry. An economical technique to identify the Nambu-Goldstone modes is look for the field whose commutator with the symmetry generator  $N$  has non-vanishing ground

state expectation value [17]. In this simple case, since  $[N, \psi] = -\psi$ , and  $\langle vac|\psi|vac\rangle \neq 0$ , it implies that the excitations of field  $\psi$  itself are the Nambu-Goldstone modes, with gapless dispersion:  $E = \vec{p}^2$ , so that  $|\vec{p}| \rightarrow 0$  implies  $E \rightarrow 0$  [4]. This simple and instructive example shows how the notions of BEC and spontaneous symmetry breaking are related.

The dynamics of the gas of photons in a Fabry-Perot cavity (in paraxial approximation), which was realised in the experiment of Klaers *et. al.*, is governed by a complex scalar field  $\phi$ , defined in two spatial dimensions [6, 18]. The Hamiltonian for the system reads:

$$H = \int d^2x \left( \frac{1}{2m} |\vec{\nabla}\phi|^2 + g|\phi|^4 - \mu|\phi|^2 \right). \quad (2)$$

Note that the photon dispersion  $w(p)$  is quadratic:  $w = \text{constant} \times \vec{p}^2$ . It was found that the interaction of nonlinear medium with the photons was weak, and was such that it led to an effective weak photon-photon interaction, as depicted above. The chemical potential for photons is denoted by  $\mu$ . This Hamiltonian also appears in context of superfluidity and its dynamics is well studied [3]. It was observed that below a certain critical temperature, this system of weakly interacting photon gas, undergoes a phase transition to form a Bose-Einstein condensate of photons [6, 18]. In such case, the ground state average of field  $\phi$  becomes non-zero, in concurrence with photon number density average being non-zero:  $\langle \phi^\dagger \phi \rangle = \mu/g \neq 0$ . The ground state average of field  $\phi$  being non-zero, implies that the ground state is an eigenstate of destruction operator  $a_p$ :  $a_p|vac\rangle = \text{constant} \times |vac\rangle$ , where  $\phi(\vec{x}, 0) = \int \frac{d^2p}{2\pi} e^{-i\vec{p}\cdot\vec{x}} a_p$ . This means that, the ground state of the system is actually a coherent state [19, 20]. Akin to the earlier case, the action of total photon number operator  $N = \int dV \phi^\dagger \phi$  on the ground state in this theory is also ill-defined, since  $\langle N \rangle \rightarrow \infty$  implying that ground state of the theory is macroscopically populated due to BEC. As seen in the earlier case, in this case as well,  $N$  generates symmetry transformation:  $\phi \rightarrow \phi e^{i\theta}$ , such that  $[H, N] = 0$ . So the statement  $\langle N \rangle \rightarrow \infty$  is actually a statement of SSB of this symmetry. Since  $\langle \phi \rangle \neq 0$  implies  $\langle [N, \phi] \rangle \neq 0$ , one infers that the photon modes created by  $\phi$  are the gapless Nambu-Goldstone modes.

Thus, one sees that the phenomena of BEC, that of SSB and realisation of coherent photon states are interrelated.

## II. SUPERRADIANCE AND SPONTANEOUS SYMMETRY BREAKING

Consider a collection of  $N$  three level atoms, interacting with a single cavity mode. It is assumed that the three energy levels are equispaced. Hamiltonian of the system is given by:

$$H = a^\dagger a + \sum_{j=1}^N \left[ \epsilon S_j^z + \frac{\lambda}{\sqrt{N}} (a^\dagger S_j^- + a S_j^+) \right]. \quad (3)$$

Here,  $a$  and  $a^\dagger$  are destruction and creation operators for the photon in the cavity. The  $N$  atomic three level system are represented by  $N$  spin-one operators  $S_j^{\pm,z}$  [21]. This model is a generalisation of Dicke model, wherein a single radiation mode is interacting with a collection of two level atoms, which are represented by  $N$  spin-half operators [7, 10]. Partition function for the system is given by:

$$\mathcal{Z} = \text{Tr}(e^{-\beta H}). \quad (4)$$

This can be conveniently evaluated using coherent state basis:

$$\mathcal{Z} = \int \frac{d^2\alpha}{\pi} \langle \alpha | \text{Tr}^*(e^{-\beta H}) | \alpha \rangle, \quad (5)$$

since coherent state  $|\alpha\rangle$  is an eigenstate of  $a$ :  $a|\alpha\rangle = \alpha|\alpha\rangle$ ,  $\alpha$  being complex. The integration is to be done over real and imaginary parts of  $\alpha$ . Here  $\text{Tr}^*$  stands for trace over spin space. Using the property of coherent state  $|\alpha\rangle$  and that of  $|s_i\rangle$ , the eigenstates of  $S_i^z$ ; above trace can be written to the leading order in  $1/N$  as:

$$\mathcal{Z} = \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} \sum_{s_1 s_2 \dots s_N} \langle s_1 s_2 \dots s_N | e^{-\beta \sum_j h_j} | s_1 s_2 \dots s_N \rangle, \quad (6)$$

where

$$h_j = \epsilon S_j^z + \frac{\lambda}{\sqrt{N}} (\alpha^* S_j^- + \alpha S_j^+). \quad (7)$$

The sum in above expression is taken over eigenvalues of  $S_j^z$ , denoted by  $s_j$ , which take values  $-1, 0$  and  $1$ . Since  $h_j$ s are independent and identical, the trace can now simply be written as:

$$\mathcal{Z} = \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} (\text{Tr } e^{-\beta h})^N. \quad (8)$$

Here,  $h$  stands for  $h = \epsilon S^z + \frac{\lambda}{\sqrt{N}} (\alpha^* S^- + \alpha S^+)$ , and trace is to be taken over the space of  $h$ . This trace can be evaluated using matrix representation for spin-one matrices to give:

$$\mathcal{Z} = \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} [1 + 2 \cosh(\beta|\mu|)]^N, \quad (9)$$

where  $\mu = \pm\epsilon\sqrt{1 + \frac{4\lambda^2}{N\epsilon^2}|\alpha|^2}$ . It may be noted that, the use of matrix representation for evaluating trace has its limitations, in particular when one is working with large spin matrices (that is  $s \gg 1$ ). In such cases, one can resort to an alternative route to evaluate trace, using the disentangling theorem [21]. Above integral over real and imaginary parts of  $\alpha$  can be converted to modulus and argument of  $\alpha$  using equality  $\alpha = re^{i\theta}$ . Further, the integrand only depends on  $r$ , so that the angular integration can be easily done to yield  $2\pi$ . The integral then reads:

$$\mathcal{Z} = 2 \int_0^\infty dr r e^{-\beta r^2} \left[ 1 + 2 \cosh \left( \beta \epsilon \sqrt{1 + \frac{4\lambda^2}{N\epsilon^2} r^2} \right) \right]^N. \quad (10)$$

Changing the integration variable to  $y = \frac{r^2}{N}$ , the partition function can be written as:

$$\mathcal{Z} = N \int_0^\infty dy e^{-N\beta y + N \ln \left[ 1 + 2 \cosh \left( \beta \epsilon \sqrt{1 + \frac{4\lambda^2}{\epsilon^2} y} \right) \right]}. \quad (11)$$

Noting that this integral is of the form  $\int_0^\infty dy e^{N\phi(y)}$ , in the thermodynamic limit  $N \rightarrow \infty$ , it can be approximately evaluated using the Laplace method, to give:

$$\mathcal{Z} = \text{constant} \times \max_{0 \leq y \leq \infty} \left( e^{-N\beta y + N \ln \left[ 1 + 2 \cosh \left( \beta \epsilon \sqrt{1 + \frac{4\lambda^2}{\epsilon^2} y} \right) \right]} \right). \quad (12)$$

Here, max stands for the maximum value of the argument. Laplace method essentially says that the largest contribution to the integral  $\int_0^\infty dy e^{N\phi(y)}$  comes from the maximum of  $\phi(y) = -\beta y + \ln \left[ 1 + 2 \cosh \left( \beta \epsilon \sqrt{1 + \frac{4\lambda^2}{\epsilon^2} y} \right) \right]$ , which occurs when say  $y = y_0$ , and is exponentially larger than any other contribution. The maxima can be found by solving for  $y_0$  from  $\phi'(y = y_0) = 0$ . This gives rise to a transcendental equation:

$$\left( \frac{\epsilon}{4\lambda^2} \right) \eta = \frac{\sinh(\beta \epsilon \eta)}{2 \cosh(\beta \epsilon \eta) + 1}, \quad (13)$$

where  $\eta = \sqrt{1 + \frac{4\lambda^2}{\epsilon^2} y}$ , and takes values from unity to infinity. From the behaviour of RHS it is clear that when  $\frac{\epsilon}{4\lambda^2} > \frac{1}{2}$ , the function does not have any maximum for any  $\eta > 1$ . This means that  $y_0 = 0$  is the maximum in such a case. So the partition function can be written as:

$$\mathcal{Z} = \text{constant} \times [1 + 2 \cosh(\beta \epsilon)]^N. \quad (14)$$

The free energy per particle  $f(\beta)$  is then found to be:

$$-\beta f(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathcal{Z} \quad (15)$$

$$= \ln [1 + 2 \cosh(\beta\epsilon)] \quad (16)$$

On the other hand, in the case when  $\frac{\epsilon}{4\lambda^2} < \frac{1}{2}$ , one finds that there exists a critical temperature  $\beta_c$  given by:

$$\frac{\epsilon}{4\lambda^2} = \frac{\sinh(\beta_c \epsilon)}{2 \cosh(\beta_c \epsilon) + 1}, \quad (17)$$

such that for  $\beta < \beta_c$ , the function  $\phi(y)$  does not have any maximum for any  $y > 0$ . Thus, in this case as well, the free energy is given by equation (16). However, if  $\beta > \beta_c$ , then there exists one and only one  $\eta_0 (> 1)$  which solves:

$$\left(\frac{\epsilon}{4\lambda^2}\right) \eta_0 = \frac{\sinh(\beta \epsilon \eta_0)}{2 \cosh(\beta \epsilon \eta_0) + 1}. \quad (18)$$

Let  $y_0$  be given by:  $y_0 = 4\lambda^2 \sigma_0^2 - \frac{\epsilon}{4\lambda^2}$ , where  $\sigma_0 = \frac{\epsilon}{4\lambda^2} \eta_0$ ; so that the free energy reads:

$$-\beta f(\beta) = \beta \left( \frac{\epsilon}{4\lambda^2} - 4\lambda^2 \sigma_0^2 \right) + \ln (1 + 2 \cosh(4\beta \lambda^2 \sigma_0)). \quad (19)$$

Here  $\sigma_0$  is to be determined by solving  $\sigma_0 = \frac{\sinh(4\beta \lambda^2 \sigma_0)}{1 + 2 \cosh(4\beta \lambda^2 \sigma_0)}$ .

So one has found that:

A) When  $\epsilon > 2\lambda^2$ : For all values of  $\beta$ , free energy is given by:

$$-\beta f(\beta) = \ln [1 + 2 \cosh(\beta\epsilon)]. \quad (20)$$

B) When  $\epsilon < 2\lambda^2$ : In this case, there exists a critical temperature  $\beta_c$  given by (17), below and above which free energy shows different behaviour:

I) When  $\beta < \beta_c$ :

$$-\beta f(\beta) = \ln [1 + 2 \cosh(\beta\epsilon)], \quad (21)$$

II) When  $\beta > \beta_c$ :

$$-\beta f(\beta) = \beta \left( \frac{\epsilon}{4\lambda^2} - 4\lambda^2 \sigma_0^2 \right) + \ln (1 + 2 \cosh(4\beta \lambda^2 \sigma_0)), \quad (22)$$

with  $\sigma_0 = \frac{\sinh(4\beta \lambda^2 \sigma_0)}{1 + 2 \cosh(4\beta \lambda^2 \sigma_0)}$ .

In case (A), when  $\epsilon > 2\lambda^2$ , the system shows uniform behaviour, with no discontinuities in free energy as a function of temperature. On the other hand, in case (B), when  $\epsilon < 2\lambda^2$ , the system undergoes a phase transition as the critical temperature  $\beta_c$  is traversed, since the free energy changes discontinuously. Interestingly, it is found that when  $\beta > \beta_c$ , average photon number in the cavity is constant:

$$\frac{\langle a^\dagger a \rangle}{N} = 4\lambda^2 \sigma_0^2 - \frac{\epsilon}{4\lambda^2}. \quad (23)$$

The average value of total  $S_z$  in this case also turns out to be a constant:

$$\frac{1}{N} \langle \sum_{i=1}^N S_z^i \rangle = -\frac{\epsilon}{2\lambda^2}. \quad (24)$$

In contrast to this, when  $\beta < \beta_c$  or when  $\epsilon > 2\lambda^2$ , it is found that:

$$\frac{\langle a^\dagger a \rangle}{N} = 0, \quad (25)$$

$$\frac{1}{N} \langle \sum_{i=1}^N S_z^i \rangle = -\frac{2 \sinh(\beta\epsilon)}{2 \cosh(\beta\epsilon) + 1}. \quad (26)$$

This clearly shows that in  $\beta > \beta_c$  phase, the photons in the cavity are in a coherent state, often called the superradiant state. Coherence property of this state can be ascertained by looking at the second order coherence function  $g_2(t)$ , at zero delay ( $t = 0$ ):

$$g_2(0) = \frac{\langle a^\dagger a \rangle^2 - \langle a^\dagger a \rangle}{\langle a^\dagger a \rangle^2}.$$

From (23) one finds that,  $g_2(0) \rightarrow 1$  (as  $N \rightarrow \infty$ ), clearly indicating that the photons are in a coherent state  $|\alpha\rangle$ :  $a|\alpha\rangle = \alpha|\alpha\rangle$ . On the other hand, when  $\beta < \beta_c$  or when  $\epsilon > 2\lambda^2$ , is it clear from above that the photons are not in a coherent state. This result is in concurrence with that observed in the case of Dicke model [10].

Defining  $\vec{S} = \sum_{i=1}^N \vec{S}_i$ , one finds that the eigenstates of  $S^2$  and  $S_z$ :

$$S^2|l, m\rangle = l(l+1)|l, m\rangle, \quad (27)$$

$$S_z|l, m\rangle = m|l, m\rangle. \quad (28)$$

span the atomic Hilbert space, where  $l = \{0, \dots, N\}$  and  $m = \{-l, \dots, 0, \dots, l\}$ . The transition probability from initial state  $|I\rangle$  to the final state  $|F\rangle$  is given by  $P = |T_{fi}|^2$ , where  $T_{fi}$  is the transition matrix element:

$$T_{fi} = \langle F | \frac{\lambda}{\sqrt{N}} a^\dagger \sum_{i=1}^N S_i^- | I \rangle. \quad (29)$$



From above discussion it is known that the final photon state is a coherent state  $|\alpha\rangle$ , so the final state can be written as  $|F\rangle = |\alpha\rangle|l, m-1\rangle$ , whereas the initial state can be written as  $|I\rangle = a|\alpha\rangle|l, m\rangle$  [22]. The transition probability now reads:

$$I = \frac{\lambda}{\sqrt{N}} |\langle\alpha|a^\dagger a|\alpha\rangle|^2 (l+m)(l-m+1). \quad (30)$$

This relation should be contrasted with the case when the initial and final photon states are, not coherent states but, photon number eigenstates, say  $|n\rangle$  and  $|n'\rangle$  respectively:

$$J = \frac{\lambda}{\sqrt{N}} (n+1)(l+m)(l-m+1)\delta_{n,n'-1}. \quad (31)$$

When  $n = 0$ , this relation gives the probability of spontaneous emission of single photon, whereas the case of  $n \neq 0$  corresponds to induced photon emission. Dicke argued that the superradiant emission takes place due to spontaneous (single photon) emission, from states  $l = N$  and  $m \sim 0$  [7]. However it is known that the coherence properties of such photon states are different from the coherent states, in particular,  $g_2(0) < 1$  for such states. This shows that spontaneous or induced (single photon) emission can not lead to coherent superradiant emission [23]. From (30) the transition probability for a single atom emitting coherent light is found by putting  $l = 1, m = 0$ :  $I_0 = \frac{2\lambda}{\sqrt{N}}|\alpha|^2$ . On the otherhand, the transition probability for a single atom spontaneously emitting single photon is given by:  $J_0 = \frac{2\lambda}{\sqrt{N}}$ . Note that (23) implies that  $|\alpha|^2 \propto N$ , which implies :  $I_0 \propto NJ_0$ . This shows that the probability of a single atom emitting coherent light is very high compared to the probability of single photon emission. Now it is clear that the most significant contribution to emission comes from the state when it is in one photon-subtracted coherent state [20] with  $m \sim 0$  and  $l \sim N$ :  $|I\rangle = a|\alpha\rangle|N, 0\rangle$ , since the transition probability for such a state goes as:  $I = N^2 I_0 \propto N^3 J_0$ . This is also corroborated by the fact that average  $m$  value per atom is found to be less than zero:  $\frac{1}{N}\langle\sum_{i=1}^N S_z^i\rangle < 0$ , independent of  $\beta$ .

In the case, when  $\beta < \beta_c$  or when  $\epsilon > 2\lambda^2$ , since  $\frac{\langle a^\dagger a \rangle}{N} = 0$ , one finds that only spontaneous or induced (single photon) emission seem to be taking place. And since average  $m$  value per atom, though less than zero, is found to be close to zero (as  $\beta\epsilon \rightarrow 0$ ), one infers that the emission does not seem to take place as significantly from states  $m \sim 0$ , as in the earlier case.

The results found above also hold qualitatively in the case wherein one considers a collection of  $N$  four level atoms interacting with a single cavity mode. The Hamiltonian depicting

such a system is same as (3), with  $S_{\pm,z}$  being spin three half operators. In the case, when  $\epsilon/\lambda^2 > 1/3$ , the free energy reads:

$$-\beta f(\beta) = \ln(\cosh(\beta\epsilon/2) + \cosh(3\beta\epsilon/2)). \quad (32)$$

When  $\epsilon/\lambda^2 < 1/3$ , one finds a critical temperature  $\beta_c$  which solves:

$$\epsilon/\lambda^2 = \frac{\sinh(\beta_c\epsilon/2) + 3\sinh(3\beta_c\epsilon/2)}{\cosh(\beta_c\epsilon/2) + \cosh(3\beta_c\epsilon/2)}, \quad (33)$$

above which the free energy is same as (33). Below the critical temperature, the free energy is given by:

$$-\beta f(\beta) = -\beta\left(\frac{\sigma_0^2\lambda^2}{4} - \frac{\epsilon^2}{4\lambda^2}\right) + \ln(\cosh(\frac{\beta\sigma_0\lambda^2}{2}) + \cosh(\frac{3\beta\sigma_0\lambda^2}{2})), \quad (34)$$

where  $\sigma_0 = \frac{\sinh(\beta\lambda^2\sigma_0/2) + 3\sinh(3\beta\lambda^2\sigma_0/2)}{\cosh(\beta\lambda^2\sigma_0/2) + \cosh(3\beta\lambda^2\sigma_0/2)}$ . From here, following the treatment of the earlier case, it is straightforward to see that, in this phase:  $\langle a^\dagger a \rangle \propto N$ , which is in concurrence with Dicke model and its three level generalisation, given by (3).

From the above discussion, one saw that the photon number density  $\langle a^\dagger a \rangle/N$  in the superradiant state is non-vanishing, implying that the system is in the Bose-Einstein condensed state, with the ground state being macroscopically populated:  $\langle a^\dagger a \rangle \propto N$ . This shows that the superradiant phase of photons is actually a Bose-Einstein condensate, with the photons being strongly interacting with the atoms. This condensation can be seen from the spontaneous symmetry breaking point of view, by noting that system defined by (3) is invariant under phase symmetry transformation:  $a \rightarrow ae^{i\theta}$ ,  $S_- \rightarrow S_-e^{i\theta}$ , which is generated by  $U = e^{i\theta K}$ , where  $K = a^\dagger a + S_z$ , and is such that  $H = U^\dagger H U$ . The system realises a ground state, wherein  $\langle a^\dagger a \rangle \propto N$ , so that  $\langle K \rangle \rightarrow \infty$ , implying spontaneous symmetry breaking of the phase symmetry generated by  $K$ . The Nambu-Goldstone mode correspondingly can be identified, using the fact that the ground state average of the Goldstone commutator  $[K, a^\dagger]$  does not vanish, since  $\langle a^\dagger \rangle \neq 0$ , implying that the mode created by  $a^\dagger$  is the Nambu-Goldstone mode [4, 16, 17]. Since the system at hand does not have any spatial dependence, the Nambu-Goldstone mode is simply required to have zero frequency [24].

The interrelation of superradiance, spontaneous symmetry breaking and BEC becomes even more transparent, if one considers the following generalisation of (3):

$$H = \sum_k \omega_k a_k^\dagger a_k + \sum_k \sum_{j=1}^N \left[ \epsilon S_j^z + \frac{\lambda}{\sqrt{N}} \left( a_k^\dagger S_j^- + a_k S_j^+ \right) \right], \quad (35)$$

where the atoms are interacting with a continuum of photon modes, with dispersion  $\omega_k$ . Note that the conserved quantity  $K = \sum_k a_k^\dagger a_k + \sum_{j=1}^N S_j^z$ , generates phase transformations on the dynamical fields  $a_k$  and  $a_k^\dagger$ . When the system realises a ground state  $|vac\rangle$ , such that,  $\langle vac|a_k|vac\rangle = \text{constant} \times \delta(k)$ , it implies BEC of photons, since photon number density is finite:  $\langle vac|\sum_k a_k^\dagger a_k|vac\rangle/V \neq 0$ , in the ground state. From above discussion one immediately sees that this condition actually implies superradiance, and leads to emission of coherent radiation. Further in the ground state where  $\langle vac|a_k|vac\rangle = \text{constant} \times \delta(k)$ , the symmetry generated by  $K$  is spontaneously broken, since,  $\langle vac|K|vac\rangle \rightarrow \infty$  in the thermodynamic limit. As in the earlier case, using the Goldstone commutator, it is easy to see that, the modes created by  $a_k^\dagger$  are the Nambu-Goldstone modes, having gapless dispersion.

In the case when the interaction of photon modes with the atoms is off-resonance, one can integrate the atomic degrees of freedom to get an effective photon-photon interaction. It can be easily seen that, in such a case, this model would essentially be same as (2).

### III. CONCLUSION

In this paper, two generalisations of Dicke model were studied, wherein the photons in the cavity are close to the resonances of the interacting three or four level atoms, leading to strong interaction between the two. It was found that below a critical temperature both the models undergo a phase transition, paving the way to superradiant phase, leading to coherent radiation emission. In both the models, it was found that, the emission of coherent radiation occurs because of SSB of the phase symmetry of the system, leading to BEC photons in the cavity. This was compared with a model describing BEC of photons, which was realised recently in experiments, in a system of weakly interacting photon gas in a cavity. Therein below a critical temperature, it was seen that SSB of the phase symmetry of the system is responsible for the BEC of photons, giving rise to coherent emission. Thus, it is found that the SSB of phase symmetry leading to BEC of photons is responsible, in all the models, for coherent radiation emission.

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- [22] Here the fact that  $K = a^\dagger a + S_z$  is conserved, that is  $[H, K] = 0$ , is utilised.
- [23] One can indeed show that the photon number states, and the coherent states are orthogonal to each other. It follows from the fact that  $|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$ , where  $|\alpha|^2 \propto N$ , which implies that  $\langle 0|\alpha\rangle \rightarrow 0$  as  $N \rightarrow \infty$ .
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